

A TANNAKIAN RECONSTRUCTION THEOREM FOR INDBANACH SPACES

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ABSTRACT. Classically, Tannaka-Krein duality allows us to reconstruct a (co)algebra from its category of representation. In this paper we present an approach that allows us to generalise this theory to the setting of Banach spaces. This leads to several interesting applications in the directions of analytic quantum groups, bounded cohomology and galois cohomology. A large portion of this paper is dedicated to such examples.

CONTENTS

0. Introduction	1
Funding	2
1. Contracting (co)products	2
2. Categories of IndBanach (co)modules	5
2.1. IndBanach modules of IndBanach algebras	5
2.2. IndBanach comodules of IndBanach coalgebras	9
2.3. Simultaneous modules and comodules	10
3. Examples	10
3.1. Comodules of a Banach coalgebra	10
3.2. Analytic gradings	11
3.3. Gradings arising from strictly convergent and overconvergent powerseries on the unit polydisk	12
3.4. Non-example: Contracting products	14
3.5. Representations of discrete groups	14
3.6. Representations of topological groups	15
3.7. Analytic Galois descent	19
References	26

0. INTRODUCTION

Classically, Tannaka-Krein duality answers the questions of whether a compact topological group (or affine group scheme as in [8], [7]) can be recovered from its category of linear representations, and of when a category (with an appropriate fibre functor) is equivalent to representations of such a group. The answer to these questions can be seen as an application of the

Barr-Beck theorem, along with the fact that a cocontinuous linear functor on the category of vector spaces must be of the form $V \otimes -$ for some space V . This second point follows from the fact that any vector space is a colimit of copies of the base field.

Unfortunately, the above is not true for the category of Banach spaces. However, the contracting category of Banach spaces does have an analogous property, and so a brief investigation of contracting colimits in Section 1 allows us to proceed as before. We also note that the category of Banach spaces is neither complete nor cocomplete, and so we instead work in its Ind completion. Using this, we deduce an analogue of Tannaka duality for IndBanach spaces in Sections 2. In Section 3 we demonstrate some examples of applications of this theory. These include a short exploration of different analytic gradings, which the authors hope will be their first steps towards defining analytic quantum groups, and conclude with the example of Galois descent for categories of IndBanach spaces. Perhaps the most fruitful example, however, involves representations of topological groups.

In [6], Bühler shows that continuous bounded cohomology of a group G comes from the derived invariants functor on a quasi-abelian category which we denote $G\text{-Mod}^{\text{iso}}$. In Section 3.6.1 we show that this is a category of coalgebras over a comonadic functor (or comodules of an IndBanach bialgebra when the group is compact). We may therefore rephrase bounded cohomology in terms of cohomology of a monoidal comonadic functor (or an IndBanach bialgebra).

A similar interpretation of continuous group cohomology as a derived functor on some topological category can be seen, for example, in [12]. It is the goal of future work by the authors to investigate whether this topological category may also be expressed as coalgebras over a comonadic functor, and whether a functor comparing these two categories gives rise to the usual comparison map from continuous bounded cohomology to continuous group cohomology.

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1. CONTRACTING (CO)PRODUCTS

Fix a complete valued field k with non-trivial valuation, either Archimedean or non-Archimedean.

Definition 1.1. Let Ban_k denote the category of k -Banach spaces and bounded linear transformations, and let $\text{Ban}_k^{\leq 1}$ denote the wide subcategory whose morphisms are bounded linear transformations of norm at most 1. If our field is non-Archimedean then Banach spaces may be defined in two

ways, depending on whether we require norms to satisfy the usual triangle inequality or the strong triangle inequality. For most of this paper we will be able to treat both of these definitions uniformly, and will refer to them as the Archimedean and non-Archimedean cases respectively when they differ.

Definition 1.2. Let $(V_i)_{i \in I}$ be a family of Banach spaces. Let us define the *contracting product* of this family as the Banach space

$$\prod_{i \in I}^{\leq 1} V_i = \{(v_i)_{i \in I} \in \times_{i \in I} V_i \mid \sup_{i \in I} \|v_i\| \leq \infty\}$$

with norm $\|(v_i)\| = \sup_{i \in I} \|v_i\|$ in both the Archimedean and non-Archimedean cases, and the *contracting coproduct* as the Banach space

$$\prod_{i \in I}^{\leq 1} V_i = \{(v_i)_{i \in I} \in \times_{i \in I} V_i \mid \sum_{i \in I} \|v_i\| \leq \infty\}$$

with norm $\|(v_i)\| = \sum_{i \in I} \|v_i\|$ in the Archimedean case and

$$\prod_{i \in I}^{\leq 1} V_i = \{(v_i)_{i \in I} \in \times_{i \in I} V_i \mid \lim_{i \in I} \|v_i\| = 0\}$$

with norm $\|(v_i)\| = \sup_{i \in I} \|v_i\|$ in the non-Archimedean case.

Proposition 1.3. *The category $\text{Ban}_k^{\leq 1}$ has small limits and colimits.*

Proof. Indeed, it has kernels and cokernels inherited from Ban_k , and it is straightforward to check that Definition 1.2 describes products and coproducts in this category. \square

Definition 1.4. These limits and colimits give objects in Ban_k . We shall refer to them as *contracting limits and colimits* respectively, and denote them by $\lim_I^{\leq 1}$ and $\text{colim}_I^{\leq 1}$.

Remark Note that filtered contracting colimits are not left exact. For example, the maps $k \rightarrow k_{\frac{1}{n}}$ are all isomorphisms in Ban_k (and bimorphisms in $\text{Ban}_k^{\leq 1}$) but taking contracting colimits over $n \geq 1$ we obtain the morphism $k \rightarrow \{0\}$.

Contracting (co)products have the following universal property in Ban_k .

Lemma 1.5. *For all collections of morphisms $\{f_i : U \rightarrow V_i\}_{i \in I}$ (respectively $\{g_i : V_i \rightarrow W\}_{i \in I}$) such that $\{\|f_i\|\}_{i \in I}$ is bounded (respectively $\{\|g_i\|\}_{i \in I}$ is bounded) by some $M > 0$, there exists a unique map $U \rightarrow \prod_{i \in I}^{\leq 1} V_i$ (respectively $\prod_{i \in I}^{\leq 1} V_i \rightarrow W$) of norm at most M such that f_i is the composite $U \rightarrow \prod_{j \in I}^{\leq 1} V_j \rightarrow V_i$ (respectively g_i is the composite $V_i \rightarrow \prod_{j \in I}^{\leq 1} V_j \rightarrow W$). That is, $\underline{\text{Hom}}(U, \prod_{i \in I}^{\leq 1} V_i) \cong \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(U, V_i)$ and $\underline{\text{Hom}}(\prod_{i \in I}^{\leq 1} V_i, W) \cong \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(V_i, W)$.*

Proof. As the valuation on our field is assumed to be non-trivial, we may take $M \in |k^\times|$ without loss of generality, so there is $\lambda \in k^\times$ with $|\lambda| = M$. Then we may rescale our family of morphisms to $\{\frac{f_i}{\lambda}\}_{i \in I}$ in $\text{Ban}_k^{\leq 1}$. By the

universal property we get a map $\phi : U \rightarrow \prod_{i \in I}^{\leq 1} V_i$ of modulus at most 1, and scaling by λ gives our desired map, $\lambda \cdot \phi$. The proof for contracting coproducts is similar. \square

Remark It follows from Lemma 1.5 that $\prod_{i \in I}^{\leq 1}$ and $\coprod_{i \in I}^{\leq 1}$ define functors from the category $\text{Ban}_k^{\leq 1, I}$, whose objects are collections $(V_i)_{i \in I}$ of Banach spaces V_i indexed by $i \in I$ and whose morphisms $\text{Hom}((V_i)_{i \in I}, (V'_i)_{i \in I})$ are the underlying elements of the Banach space $\prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(V_i, V'_i)$, to the category Ban_k of Banach spaces. Then contracting products are right adjoints to the functors $\text{Ban}_k \rightarrow \text{Ban}_k^{\leq 1, I}$ taking a Banach space V to the collection $(V)_{i \in I}$, and likewise contracting coproducts are left adjoints to these functors.

Remark Note that contracting products and contracting coproducts do not necessarily commute. For example the natural map $\coprod_{i \in \mathbb{Z}}^{\leq 1} \prod_{j \in \mathbb{Z}}^{\leq 1} k \rightarrow \prod_{j \in \mathbb{Z}}^{\leq 1} \coprod_{i \in \mathbb{Z}}^{\leq 1} k$ is not surjective, as $(\delta_{i,j})_{i,j \in \mathbb{Z}}$ is not in the image.

Definition 1.6. Let IndBan_k be the Ind completion of Ban_k . That is, IndBan_k is the category whose objects are filtered diagrams $X : I \rightarrow \text{Ban}_k$ of Banach spaces, with morphisms $\text{Hom}(X, Y) = \text{colim}_{j \in J} \lim_{i \in I} \text{Hom}(X(i), Y(j))$. We think of these diagrams as formal colimits, and hence use the notation " $\text{colim}_{i \in I} X(i)$ " for the diagram X . For a Banach space V we will often denote by " V " the object in IndBan_k represented by the constant singleton diagram at V , and sometimes just as V when there is no ambiguity.

Proposition 1.7. *The category IndBan_k is a complete and cocomplete, locally presentable, quasi-abelian category, and can be given a closed monoidal structure extending that of Ban_k by defining*

$$(\text{"colim"}_{i \in I} X_i) \hat{\otimes} (\text{"colim"}_{j \in J} Y_j) := \text{"colim"}_{i \in I, j \in J} X_i \hat{\otimes} Y_j,$$

$$\underline{\text{Hom}}(\text{"colim"}_{i \in I} X_i, \text{"colim"}_{j \in J} Y_j) := \text{colim}_{j \in J} \lim_{i \in I} \underline{\text{Hom}}(X_i, Y_j).$$

In fact, a fairly explicit construction of limits can be found in Section 1.4.1 of [11].

Remark For an account of Ind completions see [9], and more on IndBan_k can be found in [14], [3], [4] and [11] and numerous other excellent sources. A thorough exposition of quasi-abelian categories can be found in [15]. A definition of locally presentable categories and an account of results about them, including the Adjoint Functor Theorem, can be found in [1].

Definition 1.8. We extend the definition of contracting (co)products to IndBan_k as follows. For a family $(X_i)_{i \in I}$ of objects in IndBan_k , where $X_i = \text{"colim"}_{j \in J_i} V_j^{(i)}$, let

$$\prod_{i \in I}^{\leq 1} X_i = \text{"colim"}_{(j_i)_{i \in I} \in \prod_{i \in I} J_i} \prod_{i \in I}^{\leq 1} V_{j_i}^{(i)}$$

and

$$\coprod_{i \in I}^{\leq 1} X_i = \text{"colim"}_{(j_i)_{i \in I} \in \prod_{i \in I} J_i} \coprod_{i \in I}^{\leq 1} V_{j_i}^{(i)}.$$

Proposition 1.9. *With the above definition,*

$$\underline{\text{Hom}}(\prod_{i \in I}^{\leq 1} X_i, Y) \cong \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(X_i, Y)$$

and

$$\underline{\text{Hom}}(Y, \prod_{i \in I}^{\leq 1} X_i) \cong \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(Y, X_i).$$

Proof. Note that these contracting (co)products are the induced functors from $\text{Ind}(\text{Ban}_k^{\leq 1, I})$, the Ind completion of $\text{Ban}_k^{\leq 1, I}$, to IndBan_k . Then if $Y = \text{"colim"}_{l \in L} Y_l$

$$\begin{aligned} \underline{\text{Hom}}(\prod_{i \in I}^{\leq 1} X_i, Y) &\cong \lim_{(j_i)_{i \in I}} \text{"colim"}_{l \in L} \underline{\text{Hom}}(\prod_{i \in I}^{\leq 1} V_{j_i}^{(i)}, Y_l) \\ &\cong \lim_{(j_i)_{i \in I}} \text{"colim"}_{l \in L} \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(V_{j_i}^{(i)}, Y_l) \\ &\cong \lim_{(j_i)_{i \in I}} \prod_{i \in I}^{\leq 1} \text{"colim"}_{l \in L} \underline{\text{Hom}}(V_{j_i}^{(i)}, Y_l) \\ &\cong \lim_{(j_i)_{i \in I}} \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(V_{j_i}^{(i)}, Y), \end{aligned}$$

and

$$\begin{aligned} \underline{\text{Hom}}(Y, \prod_{i \in I}^{\leq 1} X_i) &\cong \lim_{l \in L} \text{"colim"}_{(j_i)_{i \in I}} \underline{\text{Hom}}(Y_l, \prod_{i \in I}^{\leq 1} V_{j_i}^{(i)}) \\ &\cong \lim_{l \in L} \text{"colim"}_{(j_i)_{i \in I}} \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(Y_l, V_{j_i}^{(i)}) \\ &\cong \lim_{l \in L} \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(Y_l, X_i). \end{aligned}$$

Since the contracting product functor to Ban_k is a right adjoint, so is the induced functor to IndBan_k , hence contracting products are continuous. The result then follows. \square

2. CATEGORIES OF INDBANACH (CO)MODULES

2.1. IndBanach modules of IndBanach algebras.

Definition 2.1. Let \mathcal{C} be a locally presentable, k -linear, quasi-abelian category and let $F : \mathcal{C} \rightarrow \text{IndBan}_k$ be a k -linear functor. We say that F is a *fibre functor* over IndBan_k if F is bicontinuous, strongly exact, faithful and reflects strict morphisms.

The following theorem tells us when when an adjoint functor exists.

Theorem 2.2 (Adjoint Functor Theorem for Locally Presentable Categories [1]). *Let $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between locally presentable categories. Then \mathcal{F} has a left adjoint precisely if it is an accessible functor and it preserves all small limits, and \mathcal{F} has a right adjoint precisely if it preserves all small colimits.*

Definition 2.3. Recall that a *monad* \mathcal{T} on a category \mathcal{D} is a functor $\mathcal{T} : \mathcal{D} \rightarrow \mathcal{D}$ equipped with natural transformations, $\mu : \mathcal{T}\mathcal{T} \Rightarrow \mathcal{T}$, the multiplication, and $\eta : \text{Id} \Rightarrow \mathcal{T}$, the unit, satisfying conditions analogous to those for an algebra. An *algebra* over \mathcal{T} is an object M of \mathcal{D} equipped with a morphism $\zeta : \mathcal{T}M \rightarrow M$ such that $\zeta \circ \mu_M = \zeta \circ T(\zeta)$ and $\zeta \circ \eta_M = \text{id}_M$. We denote by $\mathcal{D}^{\mathcal{T}}$ the category of algebras over \mathcal{T} in \mathcal{C} .

The following result is well known.

Proposition 2.4 ([5]). *Given an adjunction $\mathcal{L} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{R}$, with unit $\eta : Id_{\mathcal{C}} \Rightarrow \mathcal{R}\mathcal{L}$ and counit $\epsilon : \mathcal{L}\mathcal{R} \Rightarrow Id_{\mathcal{D}}$, the horizontal composition $\mu = id_{\mathcal{R}} * \epsilon * id_{\mathcal{L}} : \mathcal{R}\mathcal{L}\mathcal{R}\mathcal{L} \Rightarrow \mathcal{R} \circ Id_{\mathcal{D}} \circ \mathcal{L} = \mathcal{R}\mathcal{L}$ gives $\mathcal{T} = \mathcal{R}\mathcal{L}$ the structure of a monad on \mathcal{C} with unit η . Furthermore there is a natural comparison functor $\mathcal{D} \rightarrow \mathcal{C}^{\mathcal{T}}$, $D \mapsto \mathcal{R}D$ with morphism $id_{\mathcal{R}} * \epsilon : \mathcal{T}\mathcal{R}D = \mathcal{R}\mathcal{L}\mathcal{R}D \rightarrow \mathcal{R}D$.*

Definition 2.5. With notation as above, we say that \mathcal{R} is *monadic* if this comparison functor $\mathcal{D} \rightarrow \mathcal{C}^{\mathcal{T}}$ is in fact an equivalence of categories. Dually we have the notion of *coalgebras* over *comonads*, and *comonadic* functors.

Theorem 2.6 (Barr-Beck [5]). *A functor $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{C}$ is monadic if and only if*

- i) \mathcal{F} has a left adjoint \mathcal{L} ,
- ii) \mathcal{F} reflects isomorphisms,
- iii) \mathcal{D} has, and \mathcal{F} preserves, coequalisers of \mathcal{L} -split pairs.

This gives us the following Lemma.

Lemma 2.7. *Let \mathcal{C} be a locally presentable, quasi-abelian category, and let $F : \mathcal{C} \rightarrow IndBan_k$ be a fibre functor over $IndBan_k$. Then F satisfies the conditions of Barr-Beck, so \mathcal{C} is equivalent to the category of algebras of a monadic functor T in $IndBan_k$.*

Proof. By Theorem 2.2, since $IndBan_k$ is locally presentable and a fibre functor F is both continuous and cocontinuous, hence accessible, it has a left adjoint, G . Hence property (i) is satisfied. For (ii), if $f : A \rightarrow B$ is a morphism in \mathcal{C} such that Ff is an isomorphism then it fits into a strictly coexact sequence $A \xrightarrow{f} B \rightarrow \text{Coker}(f)$, the image of which under F is then also strictly coexact, so $F(\text{Coker}(f)) = 0$. A similar argument shows $F(\text{Ker}(f)) = 0$. Since F is faithful, this means that f has trivial kernel and cokernel. It then follows from the fact that F reflects strictness that f is also an isomorphism. \mathcal{C} is quasi-abelian and hence has equalisers, and so (iii) follows from the strong exactness of F . Thus, by Theorem 2.6, F is monadic and hence \mathcal{C} is equivalent to the category of algebras of $T = FG$. \square

Lemma 2.8. *With conditions as in the previous lemma, the monad T is cocontinuous.*

Proof. This follows from the fact that F is assumed to be cocontinuous and G is a left adjoint, hence also cocontinuous. \square

Lemma 2.9. *A functor $\mathcal{V} : IndBan_k \rightarrow IndBan_k$ is isomorphic to one of the form $V \hat{\otimes} -$ for an $IndBanach$ space V if and only if it is k -linear, cocontinuous and commutes with contracting coproducts.*

Proof. For a Banach space V , $V \hat{\otimes} -$ is a left adjoint on both Ban_k and $Ban_k^{\leq 1}$ hence is cocontinuous and commutes with contracting coproducts. Since contracting coproducts commute with colimits, this is also true for

any IndBanach space V . Conversely, suppose $\mathcal{V} : \text{IndBan}_k \rightarrow \text{IndBan}_k$ is cocontinuous and commutes with contracting coproducts. Since contracting cokernels are just cokernels, and since contracting colimits are built from contracting cokernels and contracting coproducts, \mathcal{V} commutes with all contracting colimits. Each IndBanach space W can be written as a colimit of Banach spaces, $W = \text{colim}_{i \in I} W^{(i)}$ and each Banach space $W^{(i)}$ can be written as a contracting colimit $W^{(i)} = \text{colim}_{j \in J_i}^{\leq 1} W_j^{(i)}$ where $W_j^{(i)} = k_{r_{i,j}}$ for $r_{i,j} > 0$ (this follows from Lemma A.39 of [4]). Here we denote by k_r the Banach space k with the norm scaled by r . The transition maps are all given by scalar multiplication, hence can pass through \mathcal{V} and between sides of the tensor products. Then

$$\begin{aligned}
\mathcal{V}(W) &\cong \mathcal{V}(\text{colim}_{i \in I} \text{colim}_{j \in J_i}^{\leq 1} W_j^{(i)}) \\
&\cong \text{colim}_{i \in I} \text{colim}_{j \in J_i}^{\leq 1} \mathcal{V}(W_j^{(i)}) \\
&\cong \text{colim}_{i \in I} \text{colim}_{j \in J_i}^{\leq 1} \mathcal{V}(k_{r_{i,j}}) \hat{\otimes} k \\
&\cong \text{colim}_{i \in I} \text{colim}_{j \in J_i}^{\leq 1} \mathcal{V}(k) \hat{\otimes} k_{r_{i,j}} \\
&\cong \text{colim}_{i \in I} \text{colim}_{j \in J_i}^{\leq 1} \mathcal{V}(k) \hat{\otimes} W_j^{(i)} \\
&\cong \mathcal{V}(k) \hat{\otimes} (\text{colim}_{i \in I} \text{colim}_{j \in J_i}^{\leq 1} W_j^{(i)}) \cong \mathcal{V}(k) \hat{\otimes} W,
\end{aligned}$$

so \mathcal{V} is isomorphic to the functor $V \hat{\otimes} -$ for $V = \mathcal{V}(k)$. \square

Hence we have the following result.

Theorem 2.10. *Let \mathcal{C} be a locally presentable, k -linear, quasi-abelian category, equipped with a fibre functor $F : \mathcal{C} \rightarrow \text{IndBan}_k$ as in Definition 2.1. Assume further that $T = FG$ commutes with contracting coproducts, where G is the left adjoint to F . Then there exists an algebra \mathcal{A} in IndBan_k such that \mathcal{C} is equivalent to the category of left \mathcal{A} modules in IndBan_k .*

Proof. By Lemma 2.7, \mathcal{C} is equivalent to the category of algebras of T in IndBan_k . By Lemma 2.8, Lemma 2.9 and our assumption that T commutes with contracting coproducts, T is isomorphic to $\mathcal{A} \hat{\otimes} -$ for $\mathcal{A} = T(k)$. Then the fact that T is a monad is equivalent to \mathcal{A} being an algebra, and the category of T algebras in IndBan_k is then just the category of \mathcal{A} modules. \square

The condition that T commutes with contracting coproducts in the above is automatically satisfied if \mathcal{C} has a good notion of contracting coproducts and F preserves them.

Corollary 2.11. *Let \mathcal{C} be a locally presentable quasi-abelian category enriched over IndBan_k , equipped with an enriched fibre functor $F : \mathcal{C} \rightarrow \text{IndBan}_k$. Suppose further that \mathcal{C} has contracting coproduct functors $\coprod_I^{\leq 1}$ from the category whose objects are collections $(A_i)_{i \in I}$ of objects A_i of \mathcal{C} indexed by $i \in I$ and whose internal morphisms are $\underline{\text{Hom}}((A_i)_{i \in I}, (B_i)_{i \in I}) = \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(A_i, B_i)$ to \mathcal{C} such that we have natural isomorphisms $F(\coprod_{i \in I}^{\leq 1} A_i) \cong \coprod_{i \in I}^{\leq 1} F A_i$ and $\underline{\text{Hom}}(\coprod_{i \in I}^{\leq 1} A_i, B) \cong \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(A_i, B)$. Then there exists an*

algebra \mathcal{A} in IndBan_k such that \mathcal{C} is equivalent to the category of left \mathcal{A} modules in IndBan_k .

Proof. We must show that T commutes with contracting coproducts. First we note that, by the Enriched Adjoint Functor Theorem (Theorem 4.84 of [10]), G is an enriched left adjoint to F . So, for all collections $(A_i)_{i \in I}$ of IndBanach spaces and all $B \in \mathcal{C}$ we have

$$\begin{aligned} \underline{\text{Hom}}(\coprod_{i \in I}^{\leq 1} GA_i, B) &\cong \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(GA_i, B) \\ &\cong \prod_{i \in I}^{\leq 1} \underline{\text{Hom}}(A_i, FB) \\ &\cong \underline{\text{Hom}}(\coprod_{i \in I}^{\leq 1} A_i, FB) \\ &\cong \underline{\text{Hom}}(G(\coprod_{i \in I}^{\leq 1} A_i), B). \end{aligned}$$

Applying $\text{Hom}(k, -)$ then gives $\text{Hom}(\coprod_{i \in I}^{\leq 1} GA_i, B) \cong \text{Hom}(G(\coprod_{i \in I}^{\leq 1} A_i), B)$. So G commute with contracting coproducts, hence so does T . \square

We may, in fact, give an alternate and perhaps more explicit description of the algebra \mathcal{A} from Theorem 2.10 and Corollary 2.11.

Definition 2.12. Let $\mathcal{F} : \mathcal{C} \rightarrow \text{IndBan}_k$ be a functor. As IndBan_k is closed, we may define the *internal natural transformations* $\underline{\text{Hom}}(\mathcal{F}, \mathcal{F})$ from \mathcal{F} to itself as the end

$$\int_{V \in \mathcal{C}} \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V) = \text{eq} \left(\prod_{V \in \mathcal{C}} \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V) \rightrightarrows \prod_{V \rightarrow V'} \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V') \right).$$

The maps $k \rightarrow \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V)$ picking out the identity in $\underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V)$ induce a unit map

$$k \rightarrow \int_{V \in \mathcal{C}} \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V) = \underline{\text{Hom}}(\mathcal{F}, \mathcal{F})$$

and the compositions $\underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V) \hat{\otimes} \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V) \rightarrow \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V)$ induce a multiplication

$$\left(\int_{V \in \mathcal{C}} \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V) \right) \hat{\otimes} \left(\int_{V \in \mathcal{C}} \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V) \right) \rightarrow \int_{V \in \mathcal{C}} \underline{\text{Hom}}(\mathcal{F}V, \mathcal{F}V)$$

from which we give $\underline{\text{Hom}}(\mathcal{F}, \mathcal{F})$ the expected IndBanach algebra structure.

Proposition 2.13. Let \mathcal{A} be an IndBanach algebra, let \mathcal{C} be the category of its IndBanach modules and let F be the forgetful functor to IndBanach spaces. Then $\mathcal{A} \cong \underline{\text{Hom}}(F, F)$ as IndBanach algebras.

Proof. \mathcal{A} naturally gives an object of \mathcal{C} , and $F \cong \underline{\text{Hom}}(\mathcal{A}, -)$. So, by the enriched Yoneda Lemma (see Section 2.4 of [10]), $\mathcal{A} \cong \underline{\text{Hom}}(F, F)$ canonically. It is clear from construction that this is an isomorphism of IndBanach algebras. \square

Remark Suppose \mathcal{C} is the category of IndBanach modules over an IndBanach algebra \mathcal{A} . Let F denote the forgetful functor to IndBan_k , G its left adjoint, and $T = FG \cong \mathcal{A} \hat{\otimes} -$. Moerdijk proves in [13] that monoidal

structures on \mathcal{C} for which F is strong monoidal correspond to comonoidal structures on T , which in turn correspond to coalgebra structures on \mathcal{A} . For any given monoidal structure on \mathcal{C} with F strong monoidal, the counit of the adjunction gives us a morphism $T(k) \rightarrow k$. The image of $\eta_V \hat{\otimes} \eta_W$ under

$$\begin{aligned} \text{Hom}(A \hat{\otimes} B, FGA \hat{\otimes} FGB) &\cong \text{Hom}(A \hat{\otimes} B, F(GA \hat{\otimes} GB)) \\ &\cong \text{Hom}(G(A \hat{\otimes} B), GA \hat{\otimes} GB). \end{aligned}$$

gives a natural transformation $G(-\hat{\otimes}-) \Rightarrow G(-) \hat{\otimes} G(-)$. Then the composite $T(-\hat{\otimes}-) \Rightarrow F(G(-) \hat{\otimes} G(-)) \cong T(-) \hat{\otimes} T(-)$ makes T comonoidal. This gives \mathcal{A} a comultiplication compatible with its multiplication, from which the monoidal structure of \mathcal{C} comes.

2.2. IndBanach comodules of IndBanach coalgebras.

Classical Tannaka-Krein duality asks when a category \mathcal{C} is a category of comodules over a coalgebra, which we aim to provide an analytic analogue of here.

Definition 2.14. Let \mathcal{C} be a locally presentable, k -linear, quasi-abelian category, and let $F : \mathcal{C} \rightarrow \text{IndBan}_k$ be a k -linear functor. We say that F is a *co-fibre functor* if it is cocontinuous, strongly exact, faithful and reflects strict morphisms.

Theorem 2.15 (Dual Barr-Beck [5]). *A functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is comonadic if and only if*

- i) \mathcal{F} has a right adjoint \mathcal{R} ,
- ii) \mathcal{F} reflects isomorphisms,
- iii) \mathcal{C} has, and \mathcal{F} preserves, equalisers of \mathcal{R} -cosplit pairs.

Lemma 2.16. *Let \mathcal{C} be a locally presentable, quasi-abelian category, and let $F : \mathcal{C} \rightarrow \text{IndBan}_k$ be a co-fibre functor over IndBan_k . Then F satisfies the dual conditions of Barr-Beck, so \mathcal{C} is equivalent to the category of coalgebras of a comonadic functor U in IndBan_k .*

Proof. The proof is entirely similar to that of Lemma 2.7. \square

Remark Since G is a right adjoint, it is not necessarily true that G or U are cocontinuous.

Theorem 2.17. *Let \mathcal{C} be a locally presentable, k -linear, quasi-abelian category, equipped with a co-fibre functor $F : \mathcal{C} \rightarrow \text{IndBan}_k$. Assume further that $U = FG$ is cocontinuous and commutes with contracting coproducts, where G is the right adjoint to F . Then there exists a coalgebra \mathcal{B} in IndBan_k such that \mathcal{C} is equivalent to the category of left \mathcal{B} comodules in IndBan_k .*

Proof. The proof is analogous to that of Theorem 2.10. \square

Remark At first sight this result is less neat than Theorem 2.10 or Corollary 2.11, as U is not automatically cocontinuous and the assumption that it commutes with contracting coproducts is not automatic from a good notion of

contracting coproducts. However, in applications, the adjoint G , and hence the comonad U , can be described explicitly and checked for (contracting) cocontinuity.

Remark Let \mathcal{C} be the category of IndBanach comodules over an IndBanach coalgebra \mathcal{B} , and let us denote by F the forgetful functor to IndBan_k , G' its right adjoint, and $U = FG' \cong \mathcal{B} \hat{\otimes} -$. As was discussed in Section 2.1, monoidal structures on \mathcal{C} for which F is strong monoidal were shown by Moerdijk in [13] to correspond directly to monoidal structures on U , which in turn correspond to algebra structures on \mathcal{B} . This correspondence is dual to the one outlined in the final Remark of Section 2.1.

2.3. Simultaneous modules and comodules.

In the case where we have both left and right adjoints G and G' as described in Theorems 2.10 and 2.17, we relate \mathcal{A} and \mathcal{B} as follows.

Proposition 2.18. *\mathcal{A} is dualisable with dual \mathcal{B} in IndBan_k .*

Proof. The adjunction gives an adjunction between T and U

$$\text{Hom}(TV, W) \cong \text{Hom}(GV, G'W) \cong \text{Hom}(V, UW).$$

Then the unit and counit of this adjunction give the duality. \square

Remark Conversely, suppose that \mathcal{A} is a dualisable IndBanach algebra with dual \mathcal{B} . Then \mathcal{B} forms an IndBanach coalgebra, and there is an adjunction as above between the functors $T = \mathcal{A} \hat{\otimes} -$ and $U = \mathcal{B} \hat{\otimes} -$. It then follows that the category of IndBanach \mathcal{A} modules and IndBanach \mathcal{B} comodules are equivalent in a way compatible with the forgetful functor.

3. EXAMPLES

The theorems introduced above may not be considered particularly beautiful given the list of conditions required. In most applications, however, these conditions are easily checked. Below are some examples to highlight this.

3.1. Comodules of a Banach coalgebra.

Proposition 3.1. *Let B be a Banach coalgebra, viewed as an IndBanach space, and let M be an IndBanach B -comodule. Then M is isomorphic to a colimit of Banach comodules of B .*

Proof. Let \mathcal{C} be the Ind completion of the category of Banach B -comodules. The forgetful functor from the category of Banach B -comodules to Ban_k induces a cocontinuous, strongly exact, faithful functor $F : \mathcal{C} \rightarrow \text{IndBan}_k$ that reflects strict morphisms. Hence F is a co-fibre functor and \mathcal{C} is equivalent to the category of coalgebras over a comonad U . The right adjoint of the forgetful functor $B \hat{\otimes} -$ from Ban_k to Banach B -comodules induces a right adjoint G to F , and FG is isomorphic to the functor $B \hat{\otimes} -$. So it follows that \mathcal{C} is equivalent to the category of B -comodules in IndBan_k , from which the proposition follows. \square

3.2. Analytic gradings.

This example is motivated slightly by the prospect of defining analytic analogues of quantum groups. In constructing the positive part of the quantum group through Nichols algebras one works with graded vector spaces. The following gives an analytic analogue of such a grading.

Definition 3.2. Let $\text{Gr}_{\mathbb{Z}}\text{IndBan}_k$ be the category of IndBanach spaces of the form $\coprod_{n \in \mathbb{Z}}^{\leq 1} M(n)$ with morphisms that preserve this grading, that is

$$\underline{\text{Hom}}_{\text{Gr}_{\mathbb{Z}}}(\coprod_{n \in \mathbb{Z}}^{\leq 1} M(n), \coprod_{n' \in \mathbb{Z}}^{\leq 1} M'(n')) = \prod_{n \in \mathbb{Z}}^{\leq 1} \underline{\text{Hom}}(M(n), M'(n)).$$

Let F be the forgetful functor to IndBan_k which maps morphisms via the usual map

$$\begin{aligned} \prod_{n \in \mathbb{Z}}^{\leq 1} \underline{\text{Hom}}(M(n), M'(n)) &\rightarrow \prod_{n \in \mathbb{Z}}^{\leq 1} \prod_{n' \in \mathbb{Z}}^{\leq 1} \underline{\text{Hom}}(M(n), M'(n')) \\ &= \underline{\text{Hom}}(\coprod_{n \in \mathbb{Z}}^{\leq 1} M(n), \coprod_{n' \in \mathbb{Z}}^{\leq 1} M'(n')). \end{aligned}$$

$\text{Gr}_{\mathbb{Z}}\text{IndBan}_k$ is monoidal, with tensor product

$$\left(\coprod_{n \in \mathbb{Z}}^{\leq 1} M(n) \right) \hat{\otimes} \left(\coprod_{n' \in \mathbb{Z}}^{\leq 1} M'(n') \right) \cong \prod_{N \in \mathbb{Z}}^{\leq 1} \left(\coprod_{n+n'=N}^{\leq 1} M(n) \hat{\otimes} M'(n') \right).$$

Proposition 3.3. $\text{Gr}_{\mathbb{Z}}\text{IndBan}_k$ is equivalent to the monoidal category of \mathcal{B} comodules in IndBan_k , where \mathcal{B} is the bialgebra $\coprod_{n \in \mathbb{Z}}^{\leq 1} k \cdot t^n$. Here, \mathcal{B} has the comultiplication $t^n \mapsto t^n \otimes t^n$, with counit $t^n \mapsto 1$, and multiplication $t^n \cdot t^{n'} = t^{n+n'}$, with unit t^0 .

Proof. Since

$$\underline{\text{Hom}}(F(\coprod_{n \in \mathbb{Z}}^{\leq 1} M(n)), X) = \prod_{n \in \mathbb{Z}}^{\leq 1} \underline{\text{Hom}}(M(n), X),$$

we see that F is left adjoint to the functor $G : \text{IndBan}_k \rightarrow \text{Gr}_{\mathbb{Z}}\text{IndBan}_k$ that takes X to the contracting coproduct $\coprod_{\mathbb{Z}}^{\leq 1} X$. Then $U = FG$ is cocontinuous and commutes with contracting colimits, so is isomorphic to the functor $\mathcal{B} \hat{\otimes} -$. It is clear that the monoidal comonadic structure on U induces the above bialgebra structure on \mathcal{B} . \square

Remark The bialgebra \mathcal{B} can be thought of as a completion of $k[t, t^{-1}]$, the bialgebra of analytic functions on the unit circle in k , whose vector space comodules are \mathbb{Z} -graded vector spaces.

Remark In fact, if Γ is any discrete group and $\text{Gr}_{\Gamma}\text{IndBan}_k$ is the category of IndBanach spaces with an analytic Γ grading, $M = \coprod_{g \in \Gamma}^{\leq 1} M(g)$, then a similar argument to the above shows the following.

Proposition 3.4. *The analogously defined category $\text{Gr}_{\Gamma}\text{IndBan}_k$ is equivalent to the monoidal category of comodules of the bialgebra $\coprod_{g \in \Gamma}^{\leq 1} k \cdot t^g$. Here we have comultiplication $t^g \mapsto t^g \otimes t^g$, counit $t^g \mapsto 1$, multiplication $t^g \cdot t^h = t^{gh}$ and unit t^e .*

Remark If we take $\Gamma = \mathbb{Z}^n$ we obtain a completion of $k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, the coalgebra of analytic functions on the unit sphere in k^n .

Remark Note that, in the above, the forgetful functor is not continuous. The product of a collection $(\coprod_{n \in \mathbb{Z}}^{\leq 1} N(n, i))_{i \in I}$ in $\text{Gr}_{\mathbb{Z}} \text{IndBan}_k$ is $\coprod_{n \in \mathbb{Z}}^{\leq 1} \prod_{i \in I} N(n, i)$ since

$$\begin{aligned} \text{Hom}_{\text{Gr}_{\mathbb{Z}}}(\coprod_{m \in \mathbb{Z}}^{\leq 1} M(m), \coprod_{n \in \mathbb{Z}}^{\leq 1} \prod_{i \in I} N(n, i)) \\ = \prod_{n \in \mathbb{Z}}^{\leq 1} \prod_{i \in I} \text{Hom}(M(n), N(n, i)) \\ = \prod_{i \in I} \prod_{n \in \mathbb{Z}}^{\leq 1} \text{Hom}(M(n), N(n, i)) \\ = \prod_{i \in I} \text{Hom}_{\text{Gr}_{\mathbb{Z}}}(\coprod_{m \in \mathbb{Z}}^{\leq 1} M(m), \coprod_{n \in \mathbb{Z}}^{\leq 1} N(n, i)), \end{aligned}$$

but it is not necessarily true that products commute with contracting coproducts in IndBan_k .

3.3. Gradings arising from strictly convergent and overconvergent powerseries on the unit polydisk.

In the previous example, we showed that analytically \mathbb{Z}^n -graded IndBan spaces are comodules over the bialgebra of analytic functions on the unit sphere in k^n . There are, of course, other spaces of analytic functions, and these give rise to other analytic gradings.

Definition 3.5. Let $\text{Gr}_{\mathbb{N}^N} \text{IndBan}_k$ be the category whose objects are IndBan spaces of the form $\coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} M(\underline{n})$ with morphisms that respect the grading

$$\text{Hom}_{\text{Gr}_{\mathbb{N}^N}}(\coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} M(\underline{n}), \coprod_{\underline{n}' \in \mathbb{N}^N}^{\leq 1} M'(\underline{n}')) = \prod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} \text{Hom}(M(\underline{n}), M'(\underline{n})).$$

Let us denote by F the forgetful functor to IndBan_k . $\text{Gr}_{\underline{r}} \text{IndBan}_k$ is monoidal, with

$$\left(\coprod_{\underline{n}}^{\leq 1} M(\underline{n}) \right) \hat{\otimes} \left(\coprod_{\underline{n}'}^{\leq 1} M'(\underline{n}') \right) = \coprod_{\underline{n}}^{\leq 1} \left(\coprod_{\underline{m} + \underline{m}' = \underline{n}}^{\leq 1} M(\underline{m}) \hat{\otimes} M'(\underline{m}') \right).$$

Proposition 3.6. *The category $\text{Gr}_{\mathbb{N}^N} \text{IndBan}_k$ is equivalent to the category of $k\{\underline{t}\} = k\{t_1, \dots, t_N\} := \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} k \cdot \underline{t}^{\underline{n}}$ comodules, where the comultiplication maps $\underline{t}^{\underline{n}} \mapsto \underline{t}^{\underline{n}} \otimes \underline{t}^{\underline{n}}$ and the counit is $\underline{t}^{\underline{n}} \mapsto 1$, and the multiplication maps $\underline{t}^{\underline{m}} \otimes \underline{t}^{\underline{n}} \mapsto \underline{t}^{\underline{m} + \underline{n}}$ with unit $\underline{t}^{\underline{0}}$.*

Proof. This is just a variation of Proposition 3.3. □

Remark This is the bialgebra of strictly convergent powerseries on the polydisk of radius 1, $\{\underline{a} = (a_1, \dots, a_N) \in k^N \mid |a_i| \leq 1\}$. Note that strictly convergent powerseries on a polydisk of polyradius \underline{r} does not have a well defined

comultiplication unless all $r_i \leq 1$, and the counit is only well defined if all $r_i \geq 1$, hence we are restricted to the unit polydisk.

Definition 3.7. Let $\text{Gr}_{\mathbb{N}^N}^\dagger \text{IndBan}_k$ be the category whose objects are Ind-Banach spaces of the form $M = \text{"colim"}_{\underline{r} > 1} \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} M(\underline{n})_{\underline{r}^{\underline{n}}}$, with morphisms

$$\text{Hom}_{\text{Gr}_{\mathbb{N}^N}^\dagger}(M, M') = \lim_{\underline{r} > 1} \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} \text{Hom}(M(\underline{n}), M'(\underline{n}))_{(1/\underline{r})^{\underline{n}}},$$

for $M = \text{"colim"}_{\underline{r} > 1} \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} M(\underline{n})_{\underline{r}^{\underline{n}}}$ and $M' = \text{"colim"}_{\underline{r}' > 1} \coprod_{\underline{n}' \in \mathbb{N}^N}^{\leq 1} M'(\underline{n}')_{\underline{r}'^{\underline{n}'}}$. Here, limits and colimits are taken over polyradii $\underline{r} = (r_1, \dots, r_N)$ with $1 < r_i$ for $i = 1, \dots, N$, and for an IndBanach space $V = \text{"colim"}_{i \in I} V_i$ and for $\lambda \in \mathbb{R}_{>0}$, we use the notation V_λ for the IndBanach space $V_\lambda = \text{"colim"}_{i \in I} (V_i)_\lambda$, where $(V_i)_\lambda$ is the Banach space whose underlying vector space is that of V_i but with the norm scaled by λ . The category $\text{Gr}_{\mathbb{N}^N}^\dagger \text{IndBan}_k$ is monoidal, with

$$M \hat{\otimes} M' = \text{"colim"}_{\underline{r} > 1} \coprod_{\underline{n}}^{\leq 1} \left(\coprod_{\underline{m} + \underline{m}' = \underline{n}}^{\leq 1} M(\underline{m}) \hat{\otimes} M'(\underline{m}') \right)_{\underline{r}^{\underline{n}}}.$$

Proposition 3.8. *The category $\text{Gr}_{\mathbb{N}^N}^\dagger \text{IndBan}_k$ is equivalent to the monoidal category of $k\{\underline{t}\}^\dagger := \text{"colim"}_{\underline{r}' > 1} \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} k_{\underline{r}'^{\underline{n}}}$ comodules. The algebra structure comes from that of each $k\{\frac{\underline{t}}{\underline{r}}\} = \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} k_{\underline{r}^{\underline{n}}}$, whilst the counit and comultiplication are induced by the maps*

$$k\{\frac{\underline{t}}{\underline{r}}\} \rightarrow k, \quad \underline{t}^{\underline{n}} \mapsto 1, \quad k\{\frac{\underline{t}}{\underline{r}^2}\} \rightarrow k\{\frac{\underline{t}}{\underline{r}}\} \hat{\otimes} k\{\frac{\underline{t}}{\underline{r}}\}, \quad \underline{t}^{\underline{n}} \mapsto \underline{t}^{\underline{n}} \otimes \underline{t}^{\underline{n}}.$$

Proof. For each IndBanach space X

$$\begin{aligned} & \text{Hom}_{\text{Gr}_{\mathbb{N}^N}^\dagger}(\text{"colim"}_{\underline{r} > 1} \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} M(\underline{n})_{\underline{r}^{\underline{n}}}, \text{"colim"}_{\underline{r}' > \rho} \coprod_{\underline{n}' \in \mathbb{N}^N}^{\leq 1} X_{\underline{r}'^{\underline{n}}}) \\ &= \lim_{\underline{r} > 1} \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} \text{Hom}(M(\underline{n}), X)_{(1/\underline{r})^{\underline{n}}} \\ &= \lim_{\underline{r} > 1} \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} \text{Hom}(M(\underline{n})_{\underline{r}^{\underline{n}}}, X) \\ &= \text{Hom}(\text{"colim"}_{\underline{r} > 1} \coprod_{\underline{n} \in \mathbb{N}^N}^{\leq 1} M(\underline{n})_{\underline{r}^{\underline{n}}}, X) \end{aligned}$$

and so $X \mapsto \text{"colim"}_{\underline{r}' > \rho} \coprod_{\underline{n}' \in \mathbb{N}^N}^{\leq 1} X_{\underline{r}'^{\underline{n}'}}$ is right adjoint to the forgetful functor. The associated comonad is the isomorphic to $k\{\underline{t}\}^\dagger \hat{\otimes} -$. The monoidal structure on $\text{Gr}_{\mathbb{N}^N}^\dagger \text{IndBan}_k$ gives $k\{\underline{t}\}^\dagger$ the described bialgebra structure. \square

Remark $k\{\underline{t}\}^\dagger$ is referred to as the bialgebra of *overconvergent powerseries* on the polydisk of radius 1. For similar reasons to the case of strictly convergent powerseries, we are restricted to the unit polyradius.

Remark Note that we may replace \mathbb{N}^N by any monoid in the above to obtain other gradings.

3.4. Non-example: Contracting products.

Let \mathcal{C} be the category of IndBanach spaces of the form $\prod_{n \in \mathbb{Z}}^{\leq 1} M(n)$ with morphisms similar to $\text{Gr}\mathbb{Z}\text{IndBan}_K$,

$$\text{Hom}_{\mathcal{C}}(\prod_{n \in \mathbb{Z}}^{\leq 1} M(n), \prod_{n' \in \mathbb{Z}}^{\leq 1} M'(n')) = \prod_{n \in \mathbb{Z}}^{\leq 1} \text{Hom}(M(n), M'(n)),$$

and again let F be the forgetful functor to IndBan_k . Then as

$$\text{Hom}(X, F(\prod_{n \in \mathbb{Z}}^{\leq 1} M(n))) \cong \prod_{n \in \mathbb{Z}}^{\leq 1} \text{Hom}(X, M(n))$$

we see that F has as left adjoint the functor $G' : X \mapsto \prod_{n \in \mathbb{Z}}^{\leq 1} X$. However $T = FG'$ does not commute with contracting coproducts, and so is not isomorphic to taking the tensor product with an IndBanach algebra.

3.5. Representations of discrete groups.

Definition 3.9. Consider a discrete group Γ , and let $\Gamma\text{-IndBan}_k$ be the category of representations of Γ on IndBanach spaces. This has the obvious forgetful functor F to IndBan_k forgetting the action of Γ . With the diagonal action of Γ , \mathcal{C} is monoidal and F is strong monoidal.

Lemma 3.10. F has a left adjoint $G : X \mapsto \coprod_{g \in \Gamma} X$ where $h \in \Gamma$ acts on GX by mapping the copy of X indexed by g isomorphically to the copy indexed by hg .

Proof. The isomorphism $\text{Hom}(X, FY) \cong \text{Hom}_{\Gamma}(\coprod_{g \in \Gamma} X, Y)$, for X in IndBan_k and Y in $\Gamma\text{-IndBan}_k$, that gives this adjunction takes $f : X \rightarrow Y$ to the morphism defined on the copy of X indexed by g as $X \xrightarrow{f} Y \xrightarrow{g} Y$. The inverse to this map just restricts a morphism $\coprod_{g \in \Gamma} X \rightarrow Y$ to the copy of X indexed by the identity $1 \in G$. \square

Proposition 3.11. $\Gamma\text{-IndBan}_k$ is equivalent to the monoidal category of $\mathcal{A} = \coprod_{g \in \Gamma} k$ modules in IndBan_k . Here, the multiplication on \mathcal{A} is determined by mapping isomorphically the tensor product $k \hat{\otimes} k$ of the copies of k indexed by g and g' to the gg' copy of k in \mathcal{A} , with the unit being the map from k to the copy of k indexed by 1 . The comultiplication on \mathcal{A} maps the copy of k indexed by g isomorphically to the tensor product $k \hat{\otimes} k$ of the copies of k indexed by g in $\mathcal{A} \hat{\otimes} \mathcal{A}$.

Proof. This follows from Theorem 2.17, noting that $FG \cong \mathcal{A} \hat{\otimes} -$. \square

Remark Since \mathcal{A} is an essentially monomorphic object of IndBan_k , we may consider the underlying ring structure of \mathcal{A} . This is just $t^g \cdot t^{g'} = t^{gg'}$, for t^g representing the unit in the copy of k indexed by g . The comultiplication on \mathcal{A} is $t^g \mapsto t^g \hat{\otimes} t^g$, with counit $t^g \mapsto \delta_{g,1}$.

Definition 3.12. Let $\Gamma\text{-IndBan}_k^{\leq 1}$ be the full subcategory of $\Gamma\text{-IndBan}_k$ consisting of IndBanach spaces with an isometric action of Γ . By this we mean that an object V of $\Gamma\text{-IndBan}_k^{\leq 1}$ can be written as $V = \text{"colim"}_{i \in I} V_i$ where the action of $g \in \Gamma$ maps each V_i isometrically into some other $V_{i'}$. We

will continue to denote the restriction of F to $\Gamma\text{-IndBan}_k^{\leq 1}$ as F . $\Gamma\text{-IndBan}_k^{\leq 1}$ is again monoidal, and F is strong monoidal.

Lemma 3.13. *With notation as above, asking for an action of Γ on an IndBanach space V to be isometric is equivalent to asking that the action of Γ on V be bounded. That is, $\{\|g \cdot : V_i \rightarrow V_{i'}\| \mid g \in \Gamma\}$ is bounded.*

Proof. We can replace the norms on each V_i with the equivalent norm $v \mapsto \sup_{g \in \Gamma} \|gv\|$. \square

The following have proofs analogous to those of 3.10 and Proposition 3.11.

Lemma 3.14. *The forgetful functor F again has a left adjoint, $G' : X \mapsto \coprod_{g \in \Gamma}^{\leq 1} X$, where the action of Γ on $G'X$ is defined analogously to that on GX in Lemma 3.10.*

Proposition 3.15. *$\Gamma\text{-IndBan}_k^{\leq 1}$ is equivalent to \mathcal{A}' modules for the Banach bialgebra $\mathcal{A}' = \coprod_{g \in \Gamma}^{\leq 1} k$, with bialgebra structure defined similarly to \mathcal{A} .*

Remark \mathcal{A}' is often referred to as the *Banach group algebra*, denoted $l^1(\Gamma)$.

Remark Note that the forgetful functors from $\Gamma\text{-IndBan}_k$ and $\Gamma\text{-IndBan}_k^{\leq 1}$ also have right adjoints, $X \mapsto \prod_{\Gamma} X$ and $X \mapsto \prod_{\Gamma}^{\leq 1} X$, with similar Γ -actions to $G(X)$ and $G'(X)$. However these functors are not cocontinuous, so our monad is not isomorphic to tensoring with a coalgebra, unless Γ is finite. There are still natural morphisms $\coprod_{\Gamma} \prod_{\Gamma} X \rightarrow X$ and $X \rightarrow \prod_{\Gamma} \coprod_{\Gamma} X$, and $\coprod_{\Gamma}^{\leq 1} \prod_{\Gamma}^{\leq 1} X \rightarrow X$ and $X \rightarrow \prod_{\Gamma}^{\leq 1} \coprod_{\Gamma}^{\leq 1} X$, exhibiting an adjunction. If Γ is finite then $\mathcal{A} = \mathcal{A}' = l^1(\Gamma)$ is not only dualisable, with dual $l^\infty(\Gamma)$, but also nuclear.

3.6. Representations of topological groups.

Definition 3.16. For a locally compact topological group H and a Banach space V let us denote by $C(H, V)$ the topological vector space of continuous functions $H \rightarrow V$, with the topology of uniform convergence on compact subsets. If H is compact then $C(H, V)$ is a Banach space, and we will use $C^{\text{lu}}(H, V)$ to denote the closed subspace of *left uniformly convergent* functions. That is, the subspace of functions $f : H \rightarrow V$ such that, for each net $(h_\lambda)_{\lambda \in \Lambda}$ in H converging to the identity, $\sup_{x \in H} \|f(h_\lambda x) - f(x)\|$ converges to 0.

Let us fix a locally compact topological group G .

3.6.1. Topological groups with a continuous action by isometries.

Definition 3.17. Let $G\text{-Mod}^{\text{iso}}$ be the category of strongly continuous Ind-Banach G modules for which G acts by isometries. That is, the action of G on $V = \text{"colim"}_{i \in I} V_i$ is determined by continuous maps $G \rightarrow \text{Hom}(V_i, V_{i'})$ for each $i \in I$ and for some $i' \in I$ depending on each i , where $\text{Hom}(V_i, V_{i'})$ is given the strong operator topology, whose images lie in the subspace of

isometries. The diagonal action of G makes $G\text{-Mod}^{\text{iso}}$ monoidal. We denote by F the forgetful functor to IndBan_k .

Definition 3.18. For a Banach space V , let $C_b(G, V)$ be the Banach space of bounded continuous functions from G to V , and let $C_b^{\text{lu}}(G, V)$ be the closed subspace of left uniformly continuous functions. For a general IndBan space $V = \text{"colim"}_{i \in I} V_i$ we set $C_b^{\text{lu}}(G, V) = \text{"colim"}_{i \in I} C_b^{\text{lu}}(G, V_i)$.

Lemma 3.19 ([6]). *The functor $C_b^{\text{lu}}(G, -)$ is right adjoint to the forgetful functor F .*

Proof. This is proved by Bühler in [6] for Banach spaces but follows for IndBan spaces too. \square

Proposition 3.20. *$G\text{-Mod}^{\text{iso}}$ is equivalent to the category of coalgebras over the monoidal comonad $C_b^{\text{lu}}(G, -)$.*

Proof. This follows from Lemma 2.16. \square

Corollary 3.21. *In the case where G is compact, $G\text{-Mod}^{\text{iso}}$ is equivalent to the category of coalgebras over the bialgebra $C_b^{\text{lu}}(G, k)$. Here, the multiplication is pointwise, and the comultiplication is given by the composition*

$$C_b^{\text{lu}}(G, k) \xrightarrow{\Delta} C_b^{\text{lu}}(G, C_b^{\text{lu}}(G, k)) \cong C_b^{\text{lu}}(G, k) \hat{\otimes} C_b^{\text{lu}}(G, k),$$

with $\Delta(f)(g)(g') = f(gg')$.

Proof. If G is compact, $C_b^{\text{lu}}(G, -)$ is cocontinuous and commutes with contracting colimits, so is isomorphic to $C_b^{\text{lu}}(G, k) \hat{\otimes} -$ by Lemma 2.9, and $G\text{-Mod}^{\text{iso}}$ is equivalent to IndBan $C_b^{\text{lu}}(G, k)$ -comodules. Then the monoidal structure gives $C_b^{\text{lu}}(G, k)$ the usual algebra structure arising from pointwise multiplication. \square

3.6.2. *Topological Groups with a continuous action, not necessarily by isometries.*

We now consider a wider class of representations of a topological group. Suppose, for simplicity, that we can write G as a union of compact open subgroups $G = \bigcup_{i \in I} G_i$.

Definition 3.22. Let $G\text{-Mod}$ be the category of $k\text{-IndBan}$ spaces V with a *strongly continuous* action of G . By this we mean an IndBan space V such that, for each $i \in I$ there is an inductive system of Banach spaces $(V_j)_{j \in J}$ and map $J \rightarrow J$, $j \mapsto j'$, such that $V \cong \text{"colim"}_{j \in J} V_j$ and the action of G on V is induced by continuous maps $G_i \rightarrow \text{Hom}(V_j, V_{j'})$ where $\text{Hom}(V_j, V_{j'})$ is given the strong operator topology. We will denote by F the forgetful functor from $G\text{-Mod}$ to the category of IndBan spaces. The diagonal action of G makes $G\text{-Mod}$ monoidal, with trivial action on the monoidal unit k , and F is strong monoidal.

Remark If $V \in G\text{-Mod}$ is a Banach space then this just means that the action by G is strongly continuous in the usual sense.

Remark Note that $G\text{-Mod}^{\text{iso}}$ sits as a full subcategory of $G\text{-Mod}$.

Definition 3.23. Let \mathcal{T}_c be the category of locally convex topological vector spaces. Then \mathcal{T}_c is cocomplete, so there is a functor $IL : \text{IndBan}_k \rightarrow \mathcal{T}_c$ which evaluates a colimit in \mathcal{T}_c , $IL(\text{"colim"}_{j \in J} V_j) = \text{colim}_{j \in J} V_j$. It is discussed in more detail in [14].

Proposition 3.24. *An action of G on an IndBanach space V is strongly continuous if and only if the induced map $IL(V) \rightarrow C(G, IL(V))$ is continuous.*

Proof. Indeed, continuous maps $G_i \rightarrow \text{Hom}(V_j, V_{j'})$ give continuous maps $V_j \rightarrow C(G_i, V_{j'})$. Then the compositions $V_j \rightarrow C(G_i, V_{j'}) \rightarrow C(G, IL(V))$ are continuous, so induce morphisms $V_j \rightarrow \lim_{i \in I} C(G_i, IL(V)) = C(G, IL(V))$ which in turn induce the desired continuous map $IL(V) \rightarrow C(G, IL(V))$. Conversely, a continuous map $IL(V) \rightarrow C(G, IL(V))$ gives continuous maps $V_j \rightarrow C(G_i, IL(V)) = \text{colim}_{l \in J} C(G_i, V_l)$ for $i \in I$ and $j \in J$, which necessarily factor as $V_j \rightarrow C(G_i, V_{j'}) \rightarrow \text{colim}_{l \in J} C(G_i, V_l)$ as V_j is Banach. The maps $V_j \rightarrow C(G_i, V_{j'})$ then give the desired continuous maps $G_i \rightarrow \text{Hom}(V_j, V_{j'})$. \square

Definition 3.25. For any $i \in I$ and for any Banach space V , $C^{\text{lu}}(G_i, V)$ is a Banach space. For a general IndBanach space $V = \text{"colim"}_{j \in J} V_j$ we can view $C^{\text{lu}}(G_i, V)$ as the colimit $\text{"colim"}_{j \in J} C^{\text{lu}}(G_i, V_j)$ in IndBan_k , and we view $C^{\text{lu}}(G, V)$ as the limit $\lim_{i \in I} C^{\text{lu}}(G_i, V)$. $C^{\text{lu}}(G, V)$ has a left action of $g \in G$ induced by the right regular actions of G_i on $C^{\text{lu}}(G_i, V_j)$.

Lemma 3.26. $C^{\text{lu}}(G, V)$ can be expressed as the colimit of spaces

$$\left\{ (f_i)_{i \in I} \in \prod_{i \in I}^{\leq 1} C^{\text{lu}}(G_i, V_{j_i})_{r_i} \left| \begin{array}{l} \text{For all } i \leq i' \text{ there exists } j \geq j_i, j_{i'} \\ \text{with } \phi_{j_i, j} \circ f_i|_{G_{i'}} = \phi_{j_{i'}, j} \circ f_{i'} \end{array} \right. \right\}$$

indexed over pairs $((j_i)_{i \in I}, (r_i)_{i \in I})$ where $(j_i)_{i \in I}$ is a collection of indecies in J and $(r_i)_{i \in I}$ is a collection of positive real numbers, both indexed over I . Here, $\phi_{j, j'} : V_j \rightarrow V_{j'}$ are the transition maps in the inductive system $(V_j)_{j \in J}$.

Proof. Firstly, note that $C^{\text{lu}}(G, V)$ is the kernel of the map $\prod_{i \in I} C^{\text{lu}}(G_i, V) \rightarrow \prod_{(i \leq i') \in I} C^{\text{lu}}(G_{i'}, V)$ defined by $\pi_{i, i'} = \pi_{i'} - \rho_{i, i'} \circ \pi_i$ where $\pi_{i, i'}$ and π_i are the obvious projections and $\rho_{i, i'} : C^{\text{lu}}(G_{i'}, V) \rightarrow C^{\text{lu}}(G_i, V)$ is the restriction map. By the explicit description of limits in [11],

$$\prod_{i \in I} C^{\text{lu}}(G_i, V) = \text{"colim"}_{(j_i)_{i \in I}, (r_i)_{i \in I}} \prod_{i \in I}^{\leq 1} C^{\text{lu}}(G_i, V_{j_i})_{r_i}$$

and likewise

$$\prod_{(i \leq i') \in I} C^{\text{lu}}(G_{i'}, V) = \text{"colim"}_{(j_{i, i'})_{(i \leq i') \in I}, (r_{i, i'})_{(i \leq i') \in I}} \prod_{(i \leq i') \in I}^{\leq 1} C^{\text{lu}}(G_{i'}, V_{j_{i, i'}})_{r_{i, i'}}.$$

The result then follows by direct computation, again using *loc. cit.*, of this kernel. \square

Proposition 3.27. *The action of G on $C^{\text{lu}}(G, V)$ is strongly continuous for any IndBanach space V .*

Proof. Note that, for any fixed $i_0 \in I$, we may replace I with $I_{\geq i_0}$. In which case, G_{i_0} has a strongly continuous action on the spaces describes in Lemma 3.26. \square

Definition 3.28. For V in $G\text{-Mod}$ with action maps $\pi_V^{i,j,j'} : G_i \rightarrow \text{Hom}(V_j, V_{j'})$ we get a collection of bounded linear map $V_j \rightarrow C^{\text{lu}}(G_i, V_{j'})$, $v \mapsto \pi_V^{i,j,j'}(-)(v)$, where $V = \text{"colim"}_{i \in I} V_i$. These then induce morphisms $V \rightarrow C^{\text{lu}}(G_i, V)$ in IndBan_k , inducing in turn a map $\pi_V^* : V \rightarrow C^{\text{lu}}(G, V)$, the adjoint of the action.

Lemma 3.29. *The forgetful functor $F : G\text{-Mod} \rightarrow \text{IndBan}_k$ has a right adjoint $C^{\text{lu}}(G, -)$.*

Proof. For an object V of $G\text{-Mod}$, with underlying IndBanach space FV , and an IndBanach space W , there is a natural map

$$\text{Hom}_{\text{IndBan}_k}(FV, W) \rightarrow \text{Hom}_G(V, C^{\text{lu}}(G, W)),$$

taking f to the composite $V \xrightarrow{\pi_V^*} C^{\text{lu}}(G, V) \xrightarrow{f \circ -} C^{\text{lu}}(G, W)$. Given $i \in I$, the restriction of the map

$$\text{Hom}_{\text{IndBan}}(V, C^{\text{lu}}(G, W)) \rightarrow \text{Hom}_{\text{IndBan}}(V, C^{\text{lu}}(G_i, W)) \rightarrow \text{Hom}_{\text{IndBan}}(V, W)$$

to $\text{Hom}_G(V, C^{\text{lu}}(G, W))$ provides an inverse where the first arrow is induced by the restriction map $C^{\text{lu}}(G, W) \rightarrow C^{\text{lu}}(G_i, W)$ and the second arrow is induced by the map $C^{\text{lu}}(G_i, W) \rightarrow W$ that essentially evaluates a function at $1 \in G_i \subset G$ (coming from the maps $C^{\text{lu}}(G_i, W_j) \rightarrow W_j$ for $W = \text{"colim"}_{j \in J} W_j$). Hence $\text{Hom}_{\text{IndBan}_k}(V, W) \cong \text{Hom}_G(V, C^{\text{lu}}(G, W))$. \square

The following proposition then follows from Lemma 2.16.

Proposition 3.30. *$G\text{-Mod}$ is equivalent to the category of IndBanach spaces with a coaction of the comonad $C^{\text{lu}}(G, -)$.*

Remark Here, the comultiplication $\Delta_V : C^{\text{lu}}(G, V) \rightarrow C^{\text{lu}}(G, C^{\text{lu}}(G, V))$ can be thought of as $\Delta(f)(g)(g') = f(gg')$ with counit $f \mapsto f(1)$.

Corollary 3.31. *If G is compact then $G\text{-Mod}$ is equivalent to the monoidal category of IndBanach $C^{\text{lu}}(G, k)$ -comodules. Here, the multiplication on $C^{\text{lu}}(G, k)$ is pointwise.*

Proof. If G is compact, this monad is isomorphic to $C^{\text{lu}}(G, k) \hat{\otimes}_k -$. \square

Remark The above Corollary is not true if G is not assumed to be compact, and $C^{\text{lu}}(G, k)$ is not *a priori* a coalgebra.

3.7. Analytic Galois descent.

Let $K \subset L$ be two complete valued fields, let IndBan_K and IndBan_L be their respective categories of IndBanach spaces, let $\text{Hom}_K(-, -)$ and $\text{Hom}_L(-, -)$ be their morphisms, and let $\hat{\otimes}_K$ and $\hat{\otimes}_L$ be their monoidal structures. We assume throughout that L is flat over K .

Definition 3.32. Let $\text{Res}_K^L : \text{IndBan}_L \rightarrow \text{IndBan}_K$ be the restriction functor that restricts L -IndBanach spaces to K -IndBanach spaces, and let $\text{Ind}_K^L : \text{IndBan}_K \rightarrow \text{IndBan}_L$ be the induction functor $X \mapsto L \hat{\otimes}_K X$.

Lemma 3.33. Ind_K^L and Res_K^L form an adjunction, $\text{Hom}_L(L \hat{\otimes}_K X, Y) \cong \text{Hom}_K(X, Y)$, for each K -IndBanach space X and L -IndBanach space Y , thought of as also being a K -IndBanach space.

Remark From the above Lemma we obtain a monad $\text{Res}_K^L \text{Ind}_K^L \cong L \hat{\otimes}_K -$ on IndBan_K , where the resulting K -algebra structure on L is the obvious one. It is clear that the restriction functor satisfies the conditions of Barr-Beck, and so, unsurprisingly, IndBan_L is equivalent to the category of K -IndBanach spaces with an action of L .

Remark We also obtain, from this adjunction, a comonad $U = \text{Ind}_K^L \text{Res}_K^L = L \hat{\otimes}_K -$ on IndBan_L . Here the comonad structure on U has comultiplication given by the composition $L \hat{\otimes}_K Y \cong L \hat{\otimes}_K K \hat{\otimes}_K Y \rightarrow L \hat{\otimes}_K L \hat{\otimes}_K Y$, with counit given by scalar multiplication by L on each L -IndBanach space Y . This gives the following.

Proposition 3.34. IndBan_K is equivalent to objects in IndBan_L with a coaction by $U \cong L \hat{\otimes}_K -$ via the functor $X \mapsto L \hat{\otimes}_K X$ for K -IndBanach spaces X .

Proof. This follows from the proof of Lemma 2.16. □

Remark Note that this differs from the general theory outlined previously since U is not L -linear, only K -linear. Thus we introduce the following framework to deal with this.

Definition 3.35. For algebras \mathcal{R} and \mathcal{S} in IndBan_K , let us denote by $\mathcal{R}\text{-}\mathcal{S}\text{-IndBan}_K$ the category of K -IndBanach spaces with a left action by \mathcal{R} and right action by \mathcal{S} that are compatible. Then, for K -IndBanach algebras $\mathcal{R}, \mathcal{S}, \mathcal{T}$ and objects $M \in \mathcal{R}\text{-}\mathcal{S}\text{-IndBan}_K$ and $N \in \mathcal{S}\text{-}\mathcal{T}\text{-IndBan}_K$ we obtain an object $M \hat{\otimes}_{\mathcal{S}} N$ in $\mathcal{R}\text{-}\mathcal{T}\text{-IndBan}_K$ as the coequaliser of the two maps $M \hat{\otimes}_K \mathcal{S} \hat{\otimes}_K N \rightrightarrows M \hat{\otimes}_K N$. In particular, this gives $\mathcal{R}\text{-}\mathcal{R}\text{-IndBan}_K$ a monoidal structure, $\hat{\otimes}_{\mathcal{R}}$. Suppose now that \mathcal{R} and \mathcal{S} are commutative. For left \mathcal{R} modules (respectively right \mathcal{S} modules) M and N we may view $M \otimes_K N$ as a left \mathcal{R} module (resp. right \mathcal{S} module) in two ways depending on whether we act on M or N . Thus, for $M, N \in \mathcal{R}\text{-}\mathcal{S}\text{-IndBan}_K$, there are four morphisms $\mathcal{R} \hat{\otimes}_K (M \hat{\otimes}_K N) \hat{\otimes}_K \mathcal{S} \rightarrow M \hat{\otimes}_K N$. The coequaliser of these four maps, which we denote by $M \hat{\otimes}_{\mathcal{R}\text{-}\mathcal{S}} N$, has a natural left action by \mathcal{R} and right action by \mathcal{S} , hence gives an object in $\mathcal{R}\text{-}\mathcal{S}\text{-IndBan}_K$. In particular, this

gives $\mathcal{R}\text{-}\mathcal{R}\text{-IndBan}_K$ a second monoidal structure, which we shall denote by $\hat{\otimes}_{\mathcal{R}\text{-}\mathcal{R}}$.

Lemma 3.36. *A functor $\mathcal{V} : \text{IndBan}_L \rightarrow \text{IndBan}_L$ is isomorphic to one of the form $V \hat{\otimes}_L -$ for some $V \in L\text{-}L\text{-IndBan}_K$ if and only if it is K -linear, cocontinuous and commutes with contracting coproducts.*

Proof. This is entirely similar to the proof of Lemma 2.9. The main difference is that the transition maps between copies of L (with rescaled norms) no longer pass through \mathcal{V} , and as a result $V = \mathcal{V}(L)$ now has two actions of L . On the left, $\lambda \in L$ acts by $\lambda \cdot \text{id}_{\mathcal{V}(L)}$, whilst on the right λ acts by $\mathcal{V}(\lambda \cdot \text{id}_L)$. \square

Proposition 3.37. *IndBan_K is equivalent to the category of left $(L \hat{\otimes}_K L)$ -comodules in IndBan_L via the induction functor. Here, $(L \hat{\otimes}_K L)$ is not a bialgebra in IndBan_L but instead in $L\text{-}L\text{-IndBan}_K$ with respect to the monoidal structure $\hat{\otimes}_L$. The comultiplication on $(L \hat{\otimes}_K L)$ is given by $(a \otimes b) \mapsto (a \otimes 1) \otimes (1 \otimes b)$ and the counit is just multiplication in L .*

Remark In [7], Deligne refers to objects such as $(L \hat{\otimes}_K L)$ as *groupoides*, or, in this particular case, *cogebroides*.

Proposition 3.38. *With respect to the equivalence in the above Proposition, the monoidal structure of IndBan_K corresponds to the algebra structure on $(L \hat{\otimes}_K L)$ given by $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$, with unit $1 \otimes 1$. Note that this algebra structure is with respect to the tensor product $\hat{\otimes}_{L\text{-}L}$ on $L\text{-}L\text{-IndBan}_K$.*

Definition 3.39. Consider $\text{Hom}_K(L, L)$ as an object of $L\text{-}L\text{-IndBan}_K$ with left action $(\lambda \cdot f)(a) = \lambda f(a)$ and right action $(f \cdot \lambda)(a) = f(\lambda \cdot a)$ for $\lambda, a \in L$, $f \in \text{Hom}_K(L, L)$. Then composition gives $\text{Hom}_K(L, L)$ an algebra structure with respect to $\hat{\otimes}_L$.

Proposition 3.40. *We have a non-degenerate pairing*

$$\text{Hom}_K(L, L) \hat{\otimes}_L (L \hat{\otimes}_K L) \rightarrow L, \quad \langle f, a \otimes b \rangle = f(a)b,$$

of an algebra with a coalgebra. That is, with the induced pairing between $\text{Hom}_K(L, L) \hat{\otimes}_L \text{Hom}_K(L, L)$ and $(L \hat{\otimes}_K L) \hat{\otimes}_L (L \hat{\otimes}_K L)$ given by

$$\langle f \otimes f', (a \otimes b) \otimes (a' \otimes b') \rangle = \langle f \langle f', a \otimes b \rangle, a' \otimes b' \rangle = \langle f, \langle f', a \otimes b \rangle a' \otimes b' \rangle,$$

we have that $\langle f \circ g, a \otimes b \rangle = \langle f \otimes g, \Delta(a \otimes b) \rangle$.

Proof. $\langle f \circ g, a \otimes b \rangle = f(g(a))b = \langle f, (g(a) \cdot 1)1 \otimes b \rangle = \langle f \otimes g, \Delta(a \otimes b) \rangle$. \square

Definition 3.41. Let $\Delta : \text{Hom}_K(L, L) \rightarrow \text{Hom}_K(L \hat{\otimes}_K L, L)$ be the L -linear bounded map $\Delta(f)(a \otimes b) = f(ab)$. If L/K is finite then $\text{Hom}_K(L \hat{\otimes}_K L, L) \cong \text{Hom}_K(L, L) \hat{\otimes}_{L\text{-}L} \text{Hom}_K(L, L)$ and so Δ can be viewed as a form of comultiplication.

Proposition 3.42. *We can pair $\text{Hom}_K(L \hat{\otimes}_K L, L)$ with $(L \hat{\otimes}_K L) \hat{\otimes}_{L\text{-}L} (L \hat{\otimes}_K L)$, $\langle f, (a \otimes b) \otimes (a' \otimes b') \rangle = f(a \otimes a')bb'$, $f \in \text{Hom}_K(L \hat{\otimes}_K L, L)$, $a, a', b, b' \in L$. In which case $\langle \Delta(f), (a \otimes b) \otimes (a' \otimes b') \rangle = \langle f, (a \otimes b) \cdot (a' \otimes b') \rangle$.*

Remark As a bialgebra, $L \hat{\otimes}_K L$ can be thought of as dual to $\text{Hom}_K(L, L)$. Since the Galois group, $\Gamma = \Gamma_{L/K}$, sits within $\text{Hom}_K(L, L)$, we may think of $L \hat{\otimes}_K L$ as functions on the Galois group. We shall make this more precise. Since Γ is a profinite, hence compact, topological group, its strongly continuous L -IndBanach representations should fit in the framework of Section 3.6.1. Since Γ does not act L -linearly, only K -linearly, we must modify the example slightly.

Definition 3.43. Let $\Gamma\text{-Mod}_L$ be the category of L -IndBanach spaces V with a strongly continuous action on $\text{Res}_K^L(V)$ as in Definition 3.22, given by $\pi_{V,i,i'} : \Gamma \rightarrow \text{Hom}_K(V_i, V_{i'})$ for $V \cong \text{"colim"}_{i \in I} V_i$, such that $\pi_{V,i,i'}(\sigma)(\lambda v) = \sigma(\lambda)\pi_{V,i,i'}(v)$ for $\lambda \in L$, $v \in V_i$ and $\sigma \in \Gamma$. Let F be the forgetful functor to IndBan_L . The diagonal action of Γ makes $\Gamma\text{-Mod}_L$ monoidal, with F strong monoidal. Let, for a Banach space W , $\tilde{C}^{\text{lu}}(\Gamma, W)$ be the K -Banach space of left uniformly continuous functions from Γ to W extended to an L -Banach space with the twisted action $(\lambda \cdot f)(\sigma) = \sigma(\lambda)f(\sigma)$ for $\lambda \in L$ and $f \in \tilde{C}^{\text{lu}}(\Gamma, W)$. For $W = \text{"colim"}_{i \in I} W_i$ an IndBanach space we define $\tilde{C}^{\text{lu}}(\Gamma, W) = \text{"colim"}_{i \in I} \tilde{C}^{\text{lu}}(\Gamma, W_i)$.

Lemma 3.44. *The forgetful functor F has a left adjoint $\tilde{C}^{\text{lu}}(\Gamma, -)$.*

Proof. The K -linear adjoint map $\pi_V^* : V \rightarrow C^{\text{lu}}(\Gamma, V)$ extends to an L -linear map $\pi_V^* : V \rightarrow \tilde{C}^{\text{lu}}(\Gamma, V)$. The rest follows as in the proof of Lemma 3.29. \square

Proposition 3.45. *The category $\Gamma\text{-Mod}_L$ is equivalent to monoidal category of left $C^{\text{lu}}(\Gamma, L)$ -comodules in IndBan_L . Here, $C^{\text{lu}}(\Gamma, L)$ is an object of $L\text{-}L\text{-IndBan}_K$ with left action by L as described for $\tilde{C}^{\text{lu}}(\Gamma, L)$ and right action by L the usual pointwise action on $C^{\text{lu}}(\Gamma, L)$. The multiplication is pointwise, and with respect to $\hat{\otimes}_{L-L}$, and comultiplication given by the composition*

$$C^{\text{lu}}(\Gamma, L) \xrightarrow{\Delta} C^{\text{lu}}(\Gamma, C^{\text{lu}}(\Gamma, L)) \cong C^{\text{lu}}(\Gamma, L) \hat{\otimes}_L C^{\text{lu}}(\Gamma, L)$$

where $\Delta(f)(\sigma)(\tau) = f(\tau\sigma)$ for $f \in C^{\text{lu}}(\Gamma, L)$, $\sigma, \tau \in \Gamma$.

Proof. This follows from Lemma 3.44, Lemma 2.16 and Lemma 3.36. \square

Lemma 3.46. *There is a morphism $\phi : L \hat{\otimes}_K L \rightarrow C^{\text{lu}}(\Gamma, L)$, given by $\phi(a \otimes b)(\sigma) = \sigma(a)b$, that is compatible with the multiplication and comultiplication, and has norm $\|\phi\| = 1$.*

Proof. Firstly, the fact that $\phi(a \otimes b)$ is left uniformly continuous is straightforward to prove. In fact, if $(x_\lambda)_{\lambda \in \Lambda}$ is a net converging to $1 \in \Gamma$ then $\text{Sup}_{\sigma \in \Gamma} |\phi(a \otimes b)(x_\lambda \sigma) - \phi(a \otimes b)(\sigma)|$ eventually becomes constant at 0. Secondly,

$$\begin{aligned} \phi(\lambda \cdot (a \otimes b) \cdot \mu)(\sigma) &= \sigma(\lambda)\sigma(a)b\mu = \lambda \cdot (\phi(a \otimes b)(\sigma)) \cdot \mu, \\ \phi((a \otimes b)(a' \otimes b'))(\sigma) &= \sigma(a)\sigma(a')bb' = (\phi(a \otimes b) \cdot \phi(a' \otimes b'))(\sigma), \end{aligned}$$

and

$$\Delta(\phi(a \otimes b))(\sigma)(\tau) = \tau\sigma(a)b = (\sigma(a) \cdot \phi(1 \otimes b))(\tau) = (\phi(a \otimes 1) \otimes \phi(1 \otimes b))(\sigma)(\tau)$$

for $a, b, a', b', \lambda, \mu \in L$ and $\sigma, \tau \in \Gamma$. Also, in the Archimedean case,

$$|\phi(\sum_i a_i \otimes b_i)(\sigma)| = |\sum_i \sigma(a_i)b_i| \leq \sum_i |\sigma(a_i)||b_i| = \sum_i |a_i||b_i|$$

for all $a_i, b_i \in L$ and $\sigma \in \Gamma$, hence

$$\sup_{\sigma \in \Gamma} |\phi(\alpha)(\sigma)| \leq \inf \left\{ \sum_i |a_i||b_i| \mid \alpha = \sum_i a_i \otimes b_i \right\}$$

for all $\alpha \in L \hat{\otimes}_K L$. That is, $\|\phi\| \leq 1$. The non-Archimedean case is similar. The fact that $\|\phi\| = 1$ follows since ϕ preserves the unit, which is of norm 1 in both spaces. \square

Lemma 3.47. *Let L/K be an extension of complete valued fields such that the algebraic elements are dense in L . Then $L \cong \text{colim}_{K \subset L' \subset L}^{\leq 1} L'$, where this is the contracting colimit taken in Ban_K over all finite extensions $K \subset L'$ contained in L .*

Proof. We have strict monomorphisms $L' \hookrightarrow L$ for all finite extensions $K \subset L'$ contained in L . Suppose we are given a compatible collection of bounded linear maps $\{f_{L'} : L' \rightarrow V\}_{K \subset L' \subset L}$ such that $\{\|f_{L'}\|\}_{K \subset L' \subset L}$ is bounded by some $M > 0$. Then we obtain a well defined bounded linear map $f : \bigcup_{K \subset L' \subset L} L' \rightarrow V$ defined on each L' by $f_{L'}$. The compatibility of the collection $\{f_{L'}\}_{K \subset L' \subset L}$ ensures that this is well defined. By assumption, $\bigcup_{K \subset L' \subset L} L'$ is dense in L , hence we may extend f to a unique map $L \rightarrow V$ such that $f_{L'}$ is the composition $L' \hookrightarrow L \rightarrow V$. Clearly $\|f\| \leq M$. \square

Lemma 3.48. *For an extension of complete valued fields, L/K , such that the algebraic elements are dense in L , there is an isomorphism $L \hat{\otimes}_K L \cong \text{colim}_{K \subset L' \subset L}^{\leq 1} L' \hat{\otimes}_K L'$.*

Proof. This follows from Lemma 3.47. \square

Lemma 3.49. *For G a profinite group and V a Banach space, the subspace of locally constant functions is dense in $C^{lu}(G, V)$.*

Proof. Let $f : G \rightarrow V$ be a left uniformly continuous function. For a fixed $g_0 \in G$, suppose for a contradiction that the net

$$(\sup_{g \in g_0 N} \|f(g) - f(g_0)\|)_{\substack{N \trianglelefteq G \\ [G:N] < \infty}}$$

does not converge to 0. Hence there is a sequence $(g_N)_{N \trianglelefteq G}$ converging to g_0 such that $\|f(g_N) - f(g_0)\|$ does not converge to 0, which contradicts left uniform continuity of f . Thus for all $\varepsilon > 0$ there exists $N_{g_0} \trianglelefteq G$ such that $\sup_{g \in g_0 N_{g_0}} \|f(g) - f(g_0)\| < \varepsilon$. This means that, by looking at $\{N_{g_0} \mid g_0 \in G\}$ and $f(g_0) \in V$, for each $\varepsilon > 0$ there exists a cover \mathcal{U}_ε of compact open subsets which has the property that each $U \in \mathcal{U}$ has some $\lambda_U \in V$ for which $\sup_{g \in U} \|f(g) - \lambda_U\| < \varepsilon$. By compactness of G we may assume that \mathcal{U}_ε is finite, and furthermore we can take the sets in \mathcal{U}_ε to be pairwise disjoint.

We then have that the locally constant function $\sum_{U \in \mathcal{U}} \lambda_U \chi_U$ approximates f , $\|f - \sum_{U \in \mathcal{U}} \lambda_U \chi_U\| \leq \varepsilon$, in $C^{\text{lu}}(G, V)$. \square

Lemma 3.50. *Let L/K be an extension of complete valued fields such that the algebraic elements are dense in L and form a Galois extension over K . Then there is an isomorphism $\text{colim}_{H \trianglelefteq \Gamma}^{\leq 1} C^{\text{lu}}(\Gamma/H, L) \xrightarrow{\sim} C^{\text{lu}}(\Gamma, L)$, where this is the contracting colimit taken in Ban_K over all finite index normal subgroups $H \trianglelefteq \Gamma$.*

Proof. A proof similar to that of Lemma 3.47 shows that the Banach space $\text{colim}_{H \trianglelefteq \Gamma}^{\leq 1} C^{\text{lu}}(\Gamma/H, L)$ is isomorphic to the closure of $\bigcup_{H \trianglelefteq \Gamma} C^{\text{lu}}(\Gamma, L)^H$, since the image of $C^{\text{lu}}(\Gamma/H, L)$ in $C^{\text{lu}}(\Gamma, L)$ is just the H invariant subspace. It follows from the definition of the profinite topology on Γ that a function is locally constant if and only if it lies in one of these invariant subspaces. By Lemma 3.49 this subspace is dense. \square

Lemma 3.51. *For an extension of complete valued fields, L/K , such that the algebraic elements are dense in L and form a Galois extension over K , there is an isomorphism $\text{colim}_{H \trianglelefteq \Gamma}^{\leq 1} C^{\text{lu}}(\Gamma/H, L^H) \xrightarrow{\sim} C^{\text{lu}}(\Gamma, L)$, where the contracting colimit is taken in Ban_K over all finite index normal subgroups $H \trianglelefteq \Gamma$.*

Proof. This follows from Lemma 3.47, Lemma 3.50, the fact that $C^{\text{lu}}(G, -)$ commutes with contracting colimits for finite discrete groups G , and the fact that all finite Galois extensions over K in L are of the form L^H for $H \trianglelefteq \Gamma$ of finite index. \square

Lemma 3.52. *If L/K is a finite Galois extension then the morphism ϕ in Lemma 3.46 is an isomorphism.*

Proof. By the open mapping theorem and Lemma 3.46, it is enough to show that ϕ is a bijection. First, by the Normal Basis Theorem, we may take B to be a normal basis of L over K . That is, B is a basis of L over K comprised of a single orbit of the Galois group Γ . Taking a basis $\{b \otimes 1 \mid b \in B\}$ of $L \hat{\otimes}_K L$ over L (with its right action) and the basis $\{\sigma \mapsto \delta_{\sigma, \tau} \mid \tau \in \Gamma\}$ of $C^{\text{lu}}(\Gamma, L)$ over L (with its right action) we see that ϕ is given by the matrix with entries $(\tau(b))_{(b, \tau) \in B \times \Gamma}$ indexed over $B \times \Gamma$. The columns of this matrix are all linearly independent since Γ permutes B simply transitively, hence it is invertible and so is ϕ . \square

Remark It is not clear whether ϕ is an isometry in the above finite dimensional case. This means that the norm of ϕ^{-1} might become arbitrarily large as we range over an infinite collection of such extensions. Hence ϕ may not remain an isomorphism after taking contracting colimits over infinitely many of these finite extensions (using Lemmas 3.48 and 3.51). We do, however, have the following result.

Proposition 3.53. *Let L/K be an extension of complete valued fields such that the algebraic elements, L^a , are dense in L and form a Galois extension*

over K . Then ϕ restricts to a continuous bijection between the dense subspaces $L^a \otimes_K L \subset L \hat{\otimes}_K L$ (the algebraic tensor product of L^a with L) and the subspace of locally constant functions in $C^{\text{lu}}(\Gamma, L)$.

Proof. By Lemma 3.47, there is an isomorphism

$$\text{colim}_{K \subset L' \subset L}^{\leq 1} L' \hat{\otimes}_K L \cong L \hat{\otimes}_K L$$

in IndBan_L under which the algebraic tensor product $L^a \otimes_K L$ is the union of the images of $L' \hat{\otimes}_K L = L' \otimes_K L$. By Lemma 3.50 there is an isomorphism

$$\text{colim}_{K \subset L' \subset L}^{\leq 1} C^{\text{lu}}(\Gamma_{L'/K}, L) = \text{colim}_{H \trianglelefteq \Gamma}^{\leq 1} C^{\text{lu}}(\Gamma/H, L) \cong C^{\text{lu}}(\Gamma, L)$$

under which the union of the images of $C^{\text{lu}}(\Gamma_{L'/K}, L)$ is the subspace of locally constant functions. The result then follows since ϕ restricts to the extension of the continuous bijection in Lemma 3.52 from each $L' \otimes_K L$ to the corresponding $C^{\text{lu}}(\Gamma_{L'/K}, L)$. \square

Remark The above proposition says precisely that $L \hat{\otimes}_K L$ is a completion of the space of locally constant functions with respect to a stronger topology than that inherited from $C^{\text{la}}(\Gamma, L)$. It is in this way that we may think of $L \hat{\otimes}_K L$ as functions on the Galois group Γ .

Definition 3.54. Let L/K be an extension of complete valued fields such that the algebraic elements, L^a , are dense in L and form a Galois extension over K with Galois group Γ . We think of L^a as a formal colimit over finite extensions of K in L in IndBan_K , hence as a K - IndBanach algebra. We define the IndBanach (or Bornological , following the equivalence in [2]) space of locally constant L -valued functions on G , $C^{\text{lc}}(\Gamma, L)$, to be the colimit

$$C^{\text{lc}}(\Gamma, L) = \text{"colim"}_{N \trianglelefteq \Gamma} C^{\text{lu}}(\Gamma/N, L)$$

taken over finite index normal subgroups of Γ . Similarly we define the IndBanach (or Bornological) algebraic tensor product, $L^a \otimes L$, to be the colimit

$$L^a \otimes_K L = \text{"colim"}_{K \subset L' \subset L} L' \hat{\otimes}_K L$$

taken over finite extensions L' of K in L . We may also define

$$C^{\text{lc}}(\Gamma, L^a) = \text{"colim"}_{N \trianglelefteq \Gamma} C^{\text{lu}}(\Gamma/N, L^N) = \text{"colim"}_{\substack{N \trianglelefteq \Gamma \\ K \subset L' \subset L}} C^{\text{lu}}(\Gamma/N, L')$$

and

$$L^a \otimes_K L^a = \text{"colim"}_{K \subset L' \subset L} L' \hat{\otimes}_K L' = \text{"colim"}_{\substack{K \subset L' \subset L \\ K \subset L'' \subset L}} L' \hat{\otimes}_K L''$$

in a similar way.

We may then rephrase Proposition 3.53 as the following.

Proposition 3.55. *There is a commutative diagram*

$$\begin{array}{ccc}
C^{la}(\Gamma, L) & \xrightarrow{\phi} & L \hat{\otimes}_K L \\
\uparrow & & \uparrow \\
C^{lc}(\Gamma, L) & \xrightarrow{\sim} & L^a \otimes_K L \\
\uparrow & & \uparrow \\
C^{lc}(\Gamma, L^a) & \xrightarrow{\sim} & L^a \otimes_K L^a
\end{array}$$

whose vertical arrows are bimorphisms.

Definition 3.56. Let G be a profinite group, k be a complete valued field and A be an IndBanach algebra over k . We define the IndBanach (or Bornological) Iwasawa algebra, $\Lambda_A^{\text{Born}}(G)$, to be the limit

$$\Lambda_A^{\text{Born}}(G) = \lim_{N \trianglelefteq G} A[G/N]$$

in IndBan_k taken over all open normal subgroups of G , where $A[G/N]$ is the Banach group algebra $\coprod_{G/N}^{\leq 1} A$ over A defined similarly to the algebra in Proposition 3.15. If A is a Banach algebra then we define the Banach Iwasawa algebra, $\Lambda_k^{\text{Ban}}(G)$, as the contracting limit

$$\Lambda_A^{\text{Ban}}(G) = \lim_{N \trianglelefteq G}^{\leq 1} A[G/N]$$

in Ban_k .

Proposition 3.57. Let L/K be an extension of complete valued fields such that the algebraic elements, L^a , are dense in L and form a Galois extension over K with Galois group Γ . Then, as IndBanach spaces over L , $\Lambda_L^{\text{Ban}}(\Gamma)$ is dual to $C^{la}(\Gamma, L)$, and, as L^a modules in IndBan_K , $\Lambda_{L^a}^{\text{Born}}(\Gamma)$ is dual to $C^{lc}(\Gamma, L^a) \cong L^a \otimes_K L^a$.

Proof. The first statement follows from the isomorphisms

$$\begin{aligned}
\text{Hom}_L(C^{la}(\Gamma, L), L) &= \text{Hom}_L(\text{colim}_{N \trianglelefteq \Gamma}^{\leq 1} C^{lu}(\Gamma/N, L), L) \\
&\cong \lim_{N \trianglelefteq \Gamma}^{\leq 1} \text{Hom}_L(C^{lu}(\Gamma/N, L), L) \\
&\cong \lim_{N \trianglelefteq \Gamma}^{\leq 1} \text{Hom}_L(\coprod_{\Gamma/N}^{\leq 1} L, L) \\
&\cong \lim_{N \trianglelefteq \Gamma}^{\leq 1} \coprod_{\Gamma/N}^{\leq 1} \text{Hom}_L(L, L) \\
&\cong \lim_{N \trianglelefteq \Gamma}^{\leq 1} \coprod_{\Gamma/N}^{\leq 1} L = \Lambda_L^{\text{Born}}(\Gamma).
\end{aligned}$$

The second follows from

$$\begin{aligned}
\text{Hom}_{L^a}(C^{la}(\Gamma, L^a), L^a) &= \text{Hom}_{L^a}(L^a \hat{\otimes} C^{lu}(\Gamma/N, K), L^a) \\
&\cong \text{Hom}_K(C^{lu}(\Gamma, K), L^a)
\end{aligned}$$

and a similar argument to the above. \square

Remark The above isomorphisms are not isomorphisms of algebras. The multiplications on $\Lambda_L^{\text{Ban}}(\Gamma)$ and $\Lambda_{L^a}^{\text{Born}}(\Gamma)$ induced by the respective comultiplications on $C^{lu}(\Gamma, L)$ and $C^{lc}(\Gamma, L^a)$ are twisted by the actions of Γ on L and L^a .

Definition 3.58. Let Ind_ϕ be the induction functor

$$\text{Ind}_\phi : (L \hat{\otimes}_K L)\text{-Comod} \rightarrow C^{\text{lu}}(G, k)\text{-Comod} \cong \Gamma\text{-Mod}_L, \quad M \mapsto \text{Res}_\phi M,$$

from the category of $L \hat{\otimes}_K L$ comodules in IndBan_L to $\Gamma\text{-Mod}_L$, where $\text{Ind}_\phi M$ has the same underlying IndBanach space as M but with the coaction

$$M \rightarrow (L \hat{\otimes}_K L) \hat{\otimes}_L M \xrightarrow{\phi \otimes \text{id}_M} C^{\text{lu}}(G, k) \hat{\otimes}_L M.$$

Lemma 3.59. *The induction functor Ind_ϕ is exact and faithful.*

Proof. This follows from the fact that the forgetful functors from these categories are faithful and reflect exactness, and that composition of Ind_ϕ with the forgetful functor from $\Gamma\text{-Mod}_L$ gives the forgetful functor from $(L \hat{\otimes}_K L)\text{-Comod}$. \square

Definition 3.60. Let $\Gamma\text{-Mod}_L^{\text{sm}}$ denote the essential image of Ind_ϕ in $\Gamma\text{-Mod}_L$, the category of *smooth* representations of Γ .

Proposition 3.61. *The category of smooth representations, $\Gamma\text{-Mod}_L^{\text{sm}}$, is equivalent to IndBan_K as monoidal categories via the induction functor*

$$\text{IndBan}_K \rightarrow \Gamma\text{-Mod}_L^{\text{sm}}, \quad V \mapsto L \hat{\otimes}_K V.$$

Proof. This follows from Proposition 3.37 and Lemma 3.59. \square

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